# Cryptographie

*Cours no. 3*

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# Security proofs

- What is cryptography ?
  - Cryptography's aim is to contruct schemes that achieve some goal despite the presence of an adversary.
  - Example: encryption, key-exchange, signature, electronic voting...
- Scientific approach:
  - To be rigorous, one must specify what it means to be secure.
  - Then one tries to construct schemes that achieve the desired goal, in a provable way.
  - Plain RSA encryption and signature cannot be used !

# The RSA signature scheme

- Key generation :
  - Public modulus: $N = p \cdot q$ where $p$ and $q$ are large primes.
  - Public exponent : $e$
  - Private exponent: $d$, such that $d \cdot e = 1 \mod \phi(N)$
- To sign a message $m$, the signer computes :
  - $s = m^d \mod N$
  - Only the signer can sign the message.
- To verify the signature, one checks that:
  - $m = s^e \mod N$
  - Anybody can verify the signature

# Hash-and-sign paradigm

- There are many attacks on basic RSA signatures:
  - Existential forgery: $r^e = m \mod N$
  - Chosen-message attack: $(m_1 \cdot m_2)^d = m_1^d \cdot m_2^d \mod N$
- To prevent from these attacks, one usually uses a hash function. The message is first hashed, then padded.
  - $m \longrightarrow H(m) \longrightarrow \texttt{1001}\ldots\texttt{0101}\|H(m)$
  - Example: PKCS#1 v1.5: $\mu(m) = \texttt{0001 FF....FF00}\|c_{\mathsf{SHA}}\|\mathsf{SHA}(m)$
  - ISO 9796-2: $\mu(m) = \mathsf{6A}\|m[1]\|H(m)\|\mathsf{BC}$

# Proofs for signature schemes

- Strongest security notion (Goldwasser, Micali and Rivest, 1988):
  - It must be infeasible for an adversary to forge the signature of a message, even if he can obtain the signature of messages of his choice.
- Security proof:
  - Show that from an adversary who is able to forge signature, you can solve a difficult problem, such as inverting RSA.
- Examples of provably secure signature schemes:
  - Full Domain Hash (FDH)
  - Probabilistic Signature Scheme (PSS)

# The FDH scheme

- The FDH signature scheme:
  - was designed in 1993 by Bellare and Rogaway.

$$m \longrightarrow H(m) \longrightarrow s = H(m)^d \mod N$$

  - The hash function $H(m)$ has the same output size as the modulus.
- Security of FDH
  - FDH is provably secure in the random oracle model, assuming that inverting RSA is hard.
  - In the random oracle model, the hash function is replaced by an oracle which outputs a random value for each new query.

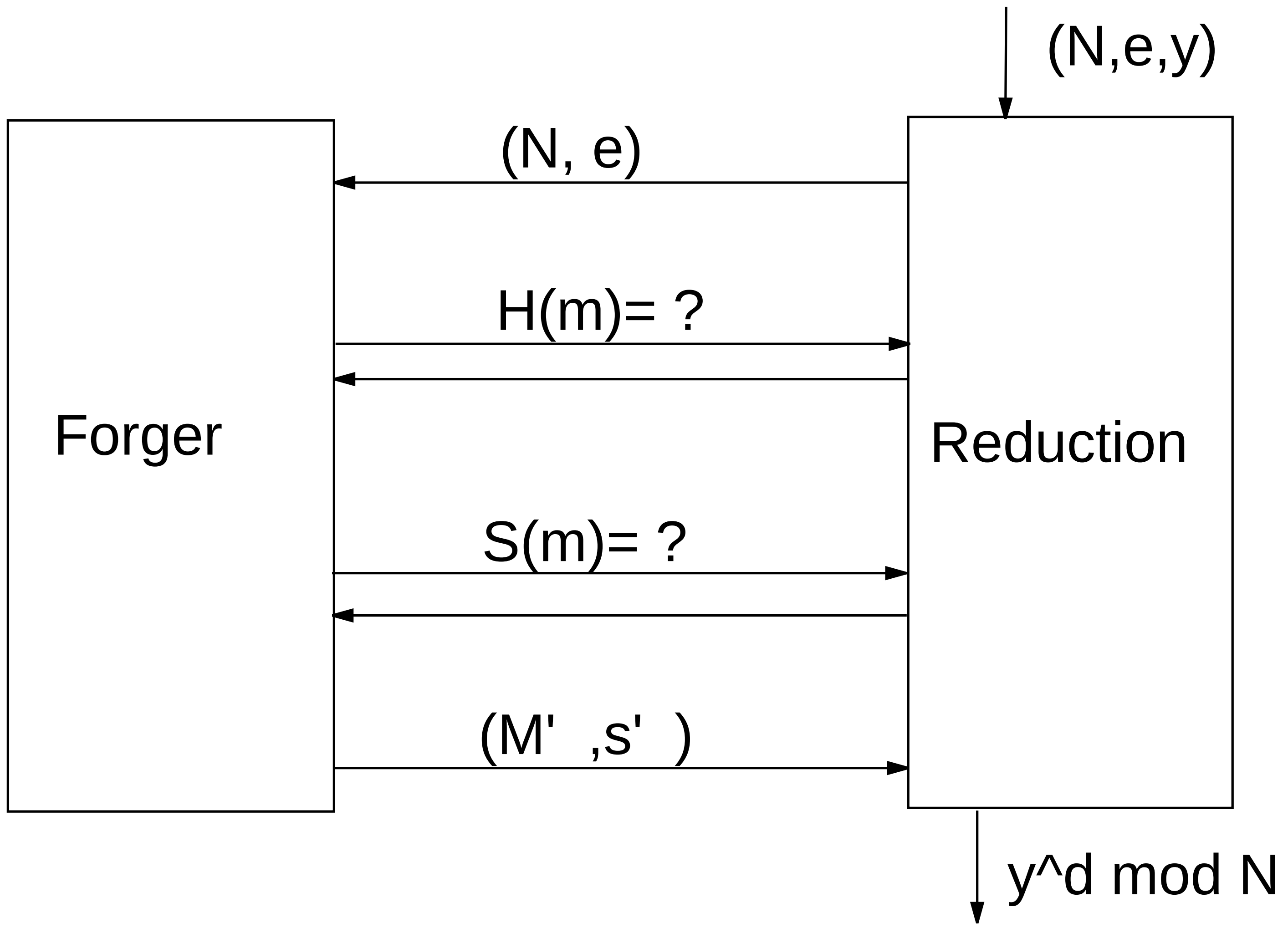
# Security proof for FDH

- We want to show that FDH is a secure signature scheme:
  - Even if the adversary requests signatures of messages of his choice, he is still unable to produce a forgery.
  - Forgery: a couple $(m', s')$ such that $s$ is a valid signature of $m$ but the signature of $m$ was never requested by the adversary.

# Security proof for FDH

- Proof in the random oracle model
  - The adversary cannot compute the hash-function by himself.
  - He must make a request to the random oracle, which answers a random, independantly distributed answer for each new query.
    - ✓ Randomly distributed in $\mathbb{Z}_N$.
- Idealized model of computation
  - A proof in the random oracle model does not imply that the scheme is secure when a concrete hash-function like SHA-1 is used.
  - Still a good guarantee.

# Security proof



# Proof of security

- We assume that there exists a succesfull adversary.
  - This adversary is an algorithm that given the public-key $(N, e)$, after at most $q_{hash}$ hash queries and $q_{sig}$ signature queries, outputs a forgery $(m', s')$.
- We will use this adversary to solve a RSA challenge: given $(N, e, y)$, output $y^d \mod N$.
  - The adversary's forgery will be used to compute $y^d \mod N$, without knowing $d$.
  - If solving such RSA challenge is assumed to be hard, then producing a forgery must be hard.

# Security proof for FDH

- Let $q_{hash}$ be the number of hash queries and $q_{sig}$ be the number of signature queries.
  - Select a random $j \in [1, q_{hash} + q_{sig} + 1]$.
- Answering a hash query for the $i$-th message $m_i$:
  - If $i \neq j$, answer $H(m_i) = r_i^e \mod N$ for random $r_i$.
  - If $i = j$, answer $H(m_j) = y$.
- Answering a signature query for $m_i$:
  - If $i \neq j$, answer $r_i = H(m_i)^d \mod N$, otherwise ($i = j$) abort.
  - We can answer all signature queries, except for message $m_j$

# Using the forgery

- Let $(m', s')$ be the forgery
  - We assume that the adversary has already made a hash query for $m'$, *i.e.* , $m' = m_i$ for some $i$.
    - Otherwise we can simulate this query.
  - Then if $i = j$, then $s' = H(m_j)^d = y^d \mod N$.
  - We return $s'$ as the solution to the RSA challenge $(N, e, y)$.

# Success probability

- Our reduction succeeds if $i = j$
  - This happens with probability $1/(q_{hash} + q_{sig} + 1)$
- From a forger that breaks FDH with probability $\varepsilon$ in time $t$, we can invert RSA with probability $\varepsilon' = \varepsilon/(q_{hash} + q_{sig} + 1)$in time $t'$ close to $t$.
- Conversely, if we assume that it is impossible to invert RSA with probability greater than $\varepsilon'$ in time $t'$, it is impossible to break FDH with probability greater than

$$\varepsilon = (q_{hash} + q_{sig} + 1) \cdot \varepsilon'$$

in time $t$ close to $t'$.

# Improving the security bound

- Instead of letting $H(m_i) = r_i^e \mod N$ for all $i \neq j$ and $H(m_j) = y$, one lets
  - $H(m_i) = r_i^e \mod N$ with probability $\alpha$
  - $H(m_i) = r_i^e \cdot y \mod N$ with probabiliy $1 - \alpha$
- Idea (published at CRYPTO 2000 by me).
  - When $H(m_i) = r_i^e \mod N$ one can answer the signature query but not use a forgery for $m_i$.
  - When $H(m_i) = r_i^e \cdot y \mod N$ one cannot answer the signature query but can use the forgery to compute $y^d \mod N$.
  - Optimize for $\alpha$.

# Improving the bound

- Probability that all signature queries are answered:
  - A signature query is answered with probability $\alpha$
  - At most $q_{sig}$ signature queries $\Rightarrow P \geq \alpha^{q_{sig}}$
- Probability that the forgery $(m_i, s')$ is useful :
  - Useful if $H(m_i) = r_i^e \cdot y \mod N$
    - ✓ $s' = H(m_i)^d = r_i \cdot y^d \mod N \Rightarrow y^d = s'/r_i \mod N$
- Global success probability :
  - $f(\alpha) = \alpha^{q_{sig}} \cdot (1 - \alpha)$
  - $f(\alpha)$ is maximum for $\alpha_m = 1 - 1/(q+1)$
  - $f(\alpha_m) \simeq 1/(e \cdot q_{sig})$ for large $q_{sig}$

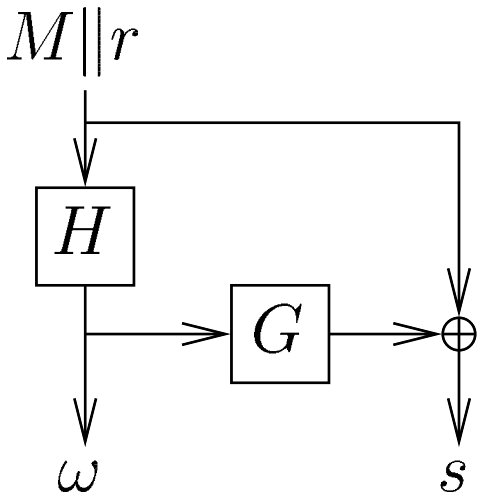
# Success probability

- From a forger that breaks FDH with probability $\varepsilon$ in time $t$, we can invert RSA with probability $\varepsilon' = \varepsilon/(4 \cdot q_{sig})$ in time $t'$ close to $t$.
- Conversely, if we assume that it is impossible to invert RSA with probability greater than $\varepsilon'$ in time $t'$, it is impossible to break FDH with probability greater than $\varepsilon = 4 \cdot q_{sig} \cdot \varepsilon'$ in time $t$ close to $t'$.
- Concrete values
  - With $q_{hash} = 2^{60}$ and $q_{sig} = 2^{30}$, we obtain $\varepsilon = 2^{32}\varepsilon'$ instead of $\varepsilon = 2^{60} \cdot \varepsilon'$
  - More secure for a given modulus size $k$.
  - A smaller modulus can be used for the same level of security: improved efficiency.

# The PSS signature cheme

- PSS (Bellare and Rogaway, Eurocrypt'96)
  - IEEE P1363a and PKCS#1 v2.1.
  - 2 variants: PSS and PSS-R (message recovery)
  - Provably secure against chosen-message attacks
  - PSS-R:

$$\mu(M,r) = \omega \| s$$



# Conclusion

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- Scientific approach:
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